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Translated by B. D.

ON MAGNETOPLASTIC FLOW

PMM Vol. 33, №6, 1969, pp. 1069-1075
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(Received January 30, 1969)

The second law of thermodynamics is used: (a) to derive the flow equations for a magnetoplastic medium; (b) to investigate in detail the magnetoplastic flow of a long thick-walled pipe; (c) to consider the flow of a pipe acted on by a nonpenetrating field; (d) to find the conditions under which the "infrozen" magnetic field facilitates plastic flow.

Magnetic fields capable of producing stresses in excess of the yield stress of metals have been achieved [1]. If the conductivity of the metal is sufficiently high, then the presence of infrozen magnetic lines of force [2] results in interaction between the plastic flow and the magnetic field. This is what constitutes magnetoplastic flow. Magnetoplastic effects are manifested if the magnetic pressure is of the order of the yield stress of the material, i.e. if $\frac{1}{8}H^2/\pi \approx k$. In the case of hard coppers ($k \approx 40$ kg/mm^2) the field intensity required is $H \approx 300$ kOe; for hard steels ($k \approx 100$ kg/mm^2) $H \approx 450$ kOe.

1. Let us make use of the second law of thermodynamics. The law of conservation of the energy W in some volume V can be written as [3]

$$dW = \delta A + d_e W \tag{1.1}$$

Here A is the work done by the external forces; $d_e W$ is the energy influx through the surface.

The work done per unit time can be resolved [4] into the work done by the external surface forces $\partial_e A / \partial t$

$$\frac{\partial_e A}{\partial t} = \oint_\Omega v_i \sigma_{ij}^\circ \, d\Omega_j \tag{1.2}$$

and that done by the external body forces (the Lorentz forces $\partial_i A / \partial t$),

$$\frac{\partial_i A}{\partial t} = \int_V \left(\frac{\mathbf{j}}{c} \times \mathbf{H}\right) \mathbf{v} \, dV \tag{1.3}$$

Moreover,

$$\frac{\partial_e W}{\partial t} = -\oint_{\Omega} \rho v_j \left(\frac{v^2}{2} + \varepsilon\right) d\Omega_j \tag{1.4}$$

Here ε is the internal energy per unit mass.

Substituting relations (1.2)–(1.4) into Eq. (1.1), we obtain

$$\frac{\partial W}{\partial t} = \oint_{\Omega} v_i \sigma_{ij}^{\circ}\, d\Omega_j + \int_V \left(\frac{\mathbf{j}}{c} \times \mathbf{H}\right) \mathbf{v}\, dV - \oint_{\Omega} \rho v_j \left(\frac{v^2}{2} + \varepsilon\right) d\Omega_j \tag{1.5}$$

On the other hand, the total energy of a body of volume V is given by

$$W = \int_V \rho \left(\frac{v^2}{2} + \varepsilon\right) dV \tag{1.6}$$

Differentiation of this relation with respect to time gives us the rate of change of energy,

$$\frac{\partial W}{\partial t} = \int_V \left[\left(\frac{v^2}{2} + \varepsilon\right) \frac{\partial \rho}{\partial t} + \rho v_i \frac{\partial v_i}{\partial t} + \rho \frac{\partial \varepsilon}{\partial t}\right] dV \tag{1.7}$$

Let us express the derivatives with respect to time in terms of the derivatives with respect to the coordinates with the aid of the mass conservation equation

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho v_j}{\partial x_j} \tag{1.8}$$

and the momentum conservation equation [2]

$$\rho \frac{\partial v_i}{\partial t} = -\rho v_j \frac{\partial v_i}{\partial x_j} + \frac{\partial \sigma_{ij}^{\circ}}{\partial x_j} + \frac{1}{4\pi}\left(H_j \frac{\partial H_i}{\partial x_j} - H_j \frac{\partial H_j}{\partial x_i}\right) \tag{1.9}$$

Substituting expressions (1.8), (1.9) into Eq. (1.7), we obtain

$$\frac{\partial W}{\partial t} = -\int_V \frac{\partial}{\partial x_j}\left[\rho v_j \left(\frac{v^2}{2} + \varepsilon\right)\right] dV + \int_V \frac{\partial}{\partial x_j}(v_i \sigma_{ij}^{\circ})\, dV - \int_V v_{ij} \sigma_{ij}^{\circ}\, dV +$$

$$+ \int_V \rho \frac{d\varepsilon}{dt}\, dV + \frac{1}{4\pi} \int_V v_i \left(H_j \frac{\partial H_i}{\partial x_j} - H_j \frac{\partial H_j}{\partial x_i}\right) dV \tag{1.10}$$

Here

$$v_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)$$

is the straining rate tensor.

Combining formulas (1.5) and (1.10), we obtain

$$\rho T \frac{dS}{dt} = v_{ij} \sigma_{ij}^{\circ} \tag{1.11}$$

(Here we have made use of the relation $d\varepsilon = TdS$ which is valid if we neglect the elastic energy.)

According to the second law of thermodynamics [5–7]

$$dS \geqslant 0$$

This implies the Duhem-Clausius condition [8]

$$v_{ij} \sigma_{ij}^{\circ} \geqslant 0 \tag{1.12}$$

If the medium is incompressible ($v_{ll} = 0$), then [9]

$$\sigma_{ik}^{\circ *} v_{ik} \geqslant 0 \quad (\sigma_{ik}^{\circ *} = \sigma_{ik}^{\circ} - {}^1/_3 \sigma_{ll}^{\circ} \delta_{ik}) \tag{1.13}$$

(Here $\sigma_{ik}^{\circ *}$ is the stress tensor deviator.)

This relation yields the flow equations in the case of an isotropic medium.

In fact, the Hamilton-Cayley equation implies that the most general relationship between the matrices v_{ik} and σ_{ik}° in the three-dimensional case is of the form [10]

$$\sigma_{ik}^{\circ} = \xi\delta_{ik} + \eta v_{ik} + \zeta v_{ij} v_{jk} \quad (\xi, \eta, \zeta \text{ are scalars}) \tag{1.14}$$

Neglecting the squares of the straining rate tensor and converting to the deviators, we obtain the flow equation for an incompressible medium,

$$\sigma_{ik}^{\circ *} = \eta v_{ik} \tag{1.15}$$

Inequality (1.13) implies that the coefficient η is positive, i.e. that $\eta > 0$.

The scalar η depends on the invariants of the stress and straining rate tensors. The various forms of this dependence are associated with specific variants of plasticity theory.

For example, let us set

$$\eta = \frac{k\sqrt{2}}{\sqrt{v_{ik}^2}} \tag{1.16}$$

(The constant k is called the yield stress and characterizes the material). We then obtain the yield equations corresponding to the Huber-Mises plasticity condition [11]

$$\tfrac{1}{2}\sigma_{ik}^{\circ *2} = k^2 \tag{1.17}$$

We note that relations (1.15) do not contain the magnetic field, i.e. that the equations of a magnetoplastic medium coincide with the yield equations of the ordinary theory of plasticity. This means that the magnetoplasticity equations postulated in [12] are incorrect.

2. Let us formulate the basic equations of magnetoplasticity. The law of conservation of momentum in a magnetoplastic medium is

$$\rho\frac{dv_i}{dt} = \frac{\partial\sigma_{ik}}{\partial x_k} \quad (\rho \text{ is the density}) \tag{2.1}$$

The stress tensor σ_{ik} is equal to the sum of the stress tensor of the material $\overset{\circ}{\sigma}_{ik}$ and the magnetic field stress tensor σ_{ik}^{H},

$$\sigma_{ik} = \sigma_{ik}^{\circ} + \sigma_{ik}^{H}, \qquad \sigma_{ik}^{H} = \frac{1}{4\pi}\left(H_i H_k - \frac{1}{2}H^2\delta_{ik}\right)$$

We can describe the magnetic field (as in magnetohydrodynamics) with the aid of the condition of infreezing of the magnetic lines of force and the condition of absence of magnetic charges,

$$\frac{\partial \mathbf{H}}{\partial t} = \operatorname{rot}(\mathbf{v}\times\mathbf{H}), \qquad \operatorname{div}\mathbf{H} = 0 \tag{2.2}$$

Let us assume that the medium is incompressible, i.e. that $v_{ll} = 0$; then $v_{ik}^{*} = v_{ik}$.

To these equations we must add flow condition (1.15) and plasticity condition (1.17).

In the case of slow motions we can neglect the derivatives with respect to time and the inertial forces, which essentially means discarding the left sides of Eqs. (2.1), (2.2). This yields the equations of steady magnetoplasticity.

3. Now let us consider the magnetoplastic flow of a long thick-walled pipe.

Making use of the cylindrical coordinate system (r, φ, z), we assume that all the quantities depend only on the coordinate r. The symmetry of the problem implies that all the nondiagonal components of the stress tensor of the material are equal to zero, i.e. that

$$\overset{\circ}{\sigma}_{r\varphi} = \overset{\circ}{\sigma}_{rz} = \overset{\circ}{\sigma}_{\varphi z} = 0 \tag{3.1}$$

We also assume that the velocity components v_φ and v_z are equal to zero,

$$v_\varphi = v_z = 0$$

From the incompressibility condition $v_{ll} = 0$ we infer that

$$v_r = \frac{\text{const}}{r} \tag{3.2}$$

This allows us to write plasticity condition (1.17) as

$$(\sigma_{rr}^\circ - \sigma_{\varphi\varphi}^\circ)^2 + (\sigma_{rr}^\circ - \sigma_{zz}^\circ)^2 + (\sigma_{\varphi\varphi}^\circ - \sigma_{zz}^\circ)^2 = 6k^2 \tag{3.3}$$

Flow condition (1.15) and the condition $v_{zz} = 0$ imply that

$$\overset{\circ\circ}{\sigma}_{zz} = 0, \quad \text{or } \overset{\circ}{\sigma}_{zz} = {}^1/_2\,(\overset{\circ}{\sigma}_{zz} + \overset{\circ}{\sigma}_{\varphi\varphi}) \tag{3.4}$$

Substituting (3.4) into (3.3), we obtain

$$\overset{\circ}{\sigma}_{rr} - \overset{\circ}{\sigma}_{\varphi\varphi} = \pm 2k \tag{3.5}$$

Discarding the left side of momentum conservation law (2.1), taking the r-th component, and making use of (3.5), we obtain

$$\frac{d\overset{\circ}{\sigma}_{rr}}{dr} \pm \frac{2k}{r} + \frac{d\sigma_{rr}^H}{dr} + \frac{\sigma_{rr}^H - \sigma_{\varphi\varphi}^H}{r} = 0 \tag{3.6}$$

4. Let us investigate the magnetoplastic flow of a pipe under various orientations of the magnetic field. Let a and b be the inside and outside radii of the pipe.

First let us consider the flow of the pipe under the internal pressure exerted by an azimuthal magnetic field. Let the infrozen magnetic field have the components $\mathbf{H}_0$ $(0, H_{\varphi 0}, H_{z0})$, and let the components of the magnetic field inside the pipe cavity (for $r \leqslant a$), which is producing the flow be $\mathbf{H}$ $(0, H_\varphi, H_{z0})$.

Magnetic field equations (2.2) (with the left side equal to zero) and expression (3.2) for the velocity give us

$$H_{\varphi 0}(r) = H_{\varphi 0}(a)\, r/a, \quad H_{z0} = \text{const} \tag{4.1}$$

where $H_{\varphi 0}(a)$ is the value of the infrozen magnetic field at the inner surface of the tube (at the inside radius). The infrozen magnetic field with this φ-component is produced by a current (of constant density over the radius) in the pipe material,

$$j_z = \frac{c}{2\pi a} H_{\varphi 0}(a) \tag{4.2}$$

(the z-component of the field can be produced by an external solenoid).

We can determine the stress distribution during flow under internal pressure by integrating Eq. (3.6) under the boundary condition

$$\overset{\circ}{\sigma}_{rr}(b) = 0 \tag{4.3}$$

In this case the magnetic field is defined by formulas (4.1), and the condition $\sigma_{\varphi\varphi}^0(b) > 0$ dictates the choice of the lower sign in Eq. (3.6) [13]. Integrating yields

$$\sigma_{rr}^\circ = -\left[2k \ln \frac{b}{r} + \frac{H_{\varphi 0}^2(a)}{4\pi a^2}(b^2 - r^2)\right] \tag{4.4}$$

We can now determine the components $\overset{\circ}{\sigma}_{\varphi\varphi}$ and $\overset{\circ}{\sigma}_{zz}$ of the stress tensor from formulas (3.4), (3.5).

The condition of continuity of the components σ_{rr} of the total stress at the inner surface of the pipe ($r = a$),

$$\sigma_{rr}^H(a - 0) = \overset{\circ}{\sigma}_{rr}(a + 0) + \delta_{rr}^H(a + 0) \tag{4.5}$$

gives us the intensity of the internal field corresponding to the onset of plastic flow,

$$\frac{H_\varphi^2}{8\pi} = 2k \ln \frac{b}{a} + \frac{H_{\varphi 0}^2(a)}{4\pi a^2}\left(b^2 - \frac{1}{2}a^2\right) \tag{4.6}$$

In the absence of an infrozen field the internal field producing plastic flow is given by the formula

$$\frac{1}{8\pi} H_\varphi^2 = 2k \ln (b / a) \tag{4.7}$$

Combining formulas (4.6) and (4.7), we see that the production of plastic flow under internal pressure requires a larger magnetic field in the presence of an infrozen field than without one (the infrozen field "hinders" plastic flow).

Now let us consider the magnetoplastic flow of a pipe under external magnetic pressure. The boundary condition in this case is

$$\sigma_{rr}^{\circ}(a) = 0 \tag{4.8}$$

Let us assume for simplicity that both the infrozen field $\mathbf{H}_0$ and the external field $\mathbf{H}$ have a φ-component only, i.e. that

$$\mathbf{H}_0(0, H_{\varphi 0}, 0), \quad \mathbf{H}(0, H_\varphi, 0)$$

The infrozen field is given by formula (4.1) as before. Integrating Eq. (3.6) under boundary condition (4.8), we obtain

$$\sigma_{rr}^{\circ} = -2k \ln \frac{r}{a} + \frac{H_{\varphi 0}^2(a)}{4\pi a^2}(r^2 - a^2) \tag{4.9}$$

(here we must choose the upper sign in Eq. (3.6) [13] in order that $\sigma_{\varphi\varphi}^{\circ} < 0$ at $r = a$).

The condition of continuity of the components σ_{rr} of the total stress at the outside surface of the pipe ($r = b$) yields the intensity of the external magnetic field corresponding to the onset of plastic flow,

$$\frac{H_\varphi^2}{8\pi} = 2k \ln \frac{b}{a} - \frac{H_{\varphi 0}^2(a)}{4\pi a^2}\left(\tfrac{1}{2} b^2 - a^2\right) \tag{4.10}$$

The magnetic field which produces plastic flow in the absence of an infrozen field is given by formula (4.7) as before. Combining Eqs. (4.10) and (4.7), we find that an infrozen magnetic field facilitates plastic flow if the pipe radii satisfy the relation

$$b > \sqrt{2}\, a \tag{4.11}$$

(i.e. if the pipe is thick enough). It is interesting to note that plastic flow can be initiated by an infrozen field alone, provided its value is given by the formula

$$\frac{H_{\varphi 0}^2(b)}{8\pi} = \frac{k \ln (b / a)}{1 - (a^2 / b^2)} \tag{4.12}$$

(This phenomenon is analogous to the pinch effect in a plasma [14].) As we see from Eq. (4.12), plastic flow under an infrozen field requires a smaller field than does flow under a nonpenetrating field if condition (4.11) is fulfilled.

Finally, let plastic flow occur in the presence of an infrozen field $\mathbf{H}_0\,(0, H_{\varphi 0}(a)\, r / / a, 0)$ and an external field $\mathbf{H}(0, H_{\varphi 0}(a) r / a, H_z)$.

The stress distribution σ_{rr} in the material is given by formula (4.9) in this case. The (external) axial magnetic field corresponding to the onset of plastic flow is given by the relation

$$\frac{H_z^2}{8\pi} = 2k \ln \frac{b}{a} - \frac{H_{\varphi 0}^2(a)}{4\pi a^2}(b^2 - a^2) \tag{4.13}$$

Thus, an infrozen magnetic field always facilitates plastic flow under these circumstances.

The authors are grateful to A. I. Akhiezer and L. I. Sedov for their valuable comments.

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Translated by A. Y.